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AUTHOR(S):

TANIKAWA, AKIO

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# THE RATE OF CONVERGENCE OF A HOMOGENEOUS MARKOV CHAIN ARISING FROM TWO-QUEUE NETWORKS

AKIO TANIKAWA (谷川明夫)

Faculty of Technology, Kanazawa University (金沢大・工)

1. Introduction. This paper is concerned with the type of random walks in two dimensions considered by Malyshev [4], which describe a wide class of two-queue systems. This type of random walks have been studied since then in many articles (e.g., [1], [2], [3], [5], [8] and [12]). These papers are mainly concerned with necessary and sufficient conditions for ergodicity, null recurrence and transience of the random walks. In the present paper, we consider the rate of convergence of the random walks when they are ergodic.

In order to define this class of random walks for our convenience, we make use of Baccelli's formulation as follows (see [1]). The state space of the random walks is the lattice in the positive quarter plane  $N \times N = \{x = (x^1, x^2): x^i, i = 1, 2, \text{ are nonnegative integers}\}$  and has to be subdivided into the following non-overlapping regions:

$D_0$  is the origin  $\{0\} \times \{0\}$ ,

$D_1$  ( $D_2$ ) denotes the open x-axis  $N^* \times \{0\}$  (or y-axis  $\{0\} \times N^*$ ),

$D_3$  denotes the open positive quarter lattice:  $N^* \times N^*$ ,

where  $N^*$  denotes the set of positive integers. There is a sequence of i.i.d.  $Z \times Z$ -valued random variables  $\{Y_n^i\}_{n \geq 1}$  associated with the region  $D_1$ : the random variable  $Y_n^i = (\xi_n^i, \eta_n^i)$  is an increment to be added at time  $n + 1$  if the current locatin is

in  $D_i$ . We restrict the range of the random variables  $Y_n^i$  in order to keep the random walk in  $N \times N$ :

the random variables  $Y_n^0$  belong to  $N \times N$ ,

the random variables  $Y_n^1$  (or  $Y_n^2$ ) to  $\{-1, 0, 1, \dots\} \times N$   
 (or  $N \times \{-1, 0, 1, \dots\}$ ),

the random variables  $Y_n^3$  to  $\{-1, 0, 1, \dots\} \times \{-1, 0, 1, \dots\}$ .

We assume that these random variables are mutually independent. We now define the random walk  $\{Z_n = (z_n^1, z_n^2)\}$  in  $N \times N$  by

$$(1.1) \quad \begin{cases} Z_0 = (z_0^1, z_0^2); \text{ where } Y = (z_0^1, z_0^2) \in N \times N \text{ is an} \\ \text{independent initial random variable,} \\ Z_{n+1} = Z_n + W_{n+1}, \quad n = 0, 1, 2, \dots \end{cases}$$

where  $W_{n+1}$  is given by

$$W_{n+1} = \sum_{i=0}^3 I(Z_n \in D_i) Y_{n+1}^i.$$

Here,  $I(B)$  denotes the indicator function of a set  $B$ . We also assume that the random walk  $\{Z_n\}$  is irreducible and aperiodic.

In the present paper, we shall obtain various rates of convergence results for the random walk (1.1) by modifying the method developed by Tweedie [11] and Thorisson [10] for one-dimensional random walk on  $R_+^1$  (see also [9] for some extension).

**2. Preliminaries.** First of all, we give necessary definitions on various rates of convergence for Markov chains. Let  $\{X_n\}$  be a temporally homogeneous Markov chain on a countable state space  $S$ . We write

$$(2.1) \quad P^n(x, A) = P(X_n \in A | X_0 = x), \quad x \in S, \quad A \subset S$$

for the  $n$ -step transition probabilities of  $\{X_n\}$ . Throughout the

paper, we suppose that  $\{X_n\}$  is irreducible and aperiodic.

We define  $\{X_n\}$  to be ergodic if, for some probability measure  $\pi$ , and every  $x \in S$ ,

$$(2.2) \quad \|P^n(x, \cdot) - \pi\| \rightarrow 0, \quad n \rightarrow \infty,$$

where  $\|\cdot\|$  denotes total variation of signed measures. Next, we define  $\{X_n\}$  to be geometrically ergodic if there exists a  $\rho < 1$  such that

$$(2.3) \quad \rho^{-n} \|P^n(x, \cdot) - \pi\| \rightarrow 0, \quad n \rightarrow \infty,$$

for every  $x \in S$ . We define sub-geometric rates of convergence as follows, which are our main concern in this paper. Let  $\Lambda_0$  denote the class of sequences  $\psi: \mathbb{N} \rightarrow \mathbb{R}_+$  which satisfy

(i)  $\psi$  is non-decreasing, with  $\psi(j) \geq 2$  for all  $j \geq 0$ .

(ii)  $\{\log \psi(j)\}/j$  is non-increasing and tends to 0 as  $j \rightarrow \infty$ .

We denote by  $\Lambda$  the class of sequences  $\psi: \mathbb{N} \rightarrow \mathbb{R}_+$  for which there exists some  $\psi_0 \in \Lambda_0$  such that

$$\liminf_{j \rightarrow \infty} \psi(j)/\psi_0(j) > 0, \quad \limsup_{j \rightarrow \infty} \psi(j)/\psi_0(j) < \infty.$$

Examples of sequences  $\psi \in \Lambda$  are

$$(2.4) \quad \psi(j) = j^\alpha (\log j)^\beta \exp(\gamma j^\delta) \vee 2$$

for  $0 < \delta < 1$  and either  $\gamma > 0$  or  $\gamma = 0$  and  $\alpha > 0$ . For  $\psi \in \Lambda$ , we call  $\{X_n\}$  ergodic of order  $\psi$  if

$$(2.5) \quad \psi(n) \|P^n(x, \cdot) - \pi\| \rightarrow 0, \quad n \rightarrow \infty,$$

for every  $x \in S$ .

We recall convenient criteria for sub-geometric rate of convergence in the following proposition, which is adapted from a more general case to the countable one.

**Proposition 1** (Theorem 2.7 of [7] and Theorem 1 (iii) of [11]).

If  $B$  is a finite subset of  $S$  and for some  $\psi \in \Lambda$ ,  $E_x[\psi^0(\tau_B)] < \infty$  for  $x \in B$  where  $\psi^0(n) = \sum_{j=1}^n \psi(j)$ , and  $E_y[\psi(\tau_B)] < \infty$  for all  $y$ , then  $\{X_n\}$  is ergodic of order  $\psi$ . Here, we denote by  $\tau_B$  the first hitting time on a set  $B$  for the Markov chain  $\{X_n\}$ .

From now on, for our convenience, we also denote by  $\Lambda_0$  the class of functions  $\psi: R_+ \rightarrow R_+$  which satisfy both (i) and (ii) for any  $t \geq 0$  instead of  $j \in N$  in the above requirements for  $\psi: N \rightarrow R_+$  of  $\Lambda_0$ . We also make use of the similar convention for  $\psi \in \Lambda$ . We now introduce subclasses of functions of  $\Lambda_0$  due to Thorisson [10] since we follow his approach in order to show sub-geometric rate of convergence results. The functions  $\psi: [0, \infty] \rightarrow [0, \infty]$  considered below are measurable, locally bounded and  $\psi(\infty) = \infty$  (i.e.,  $\lim_{t \rightarrow \infty} \psi(t) = \infty$ ). Let  $\tilde{\psi}$  be defined by  $\tilde{\psi}(t) = \int_0^t \psi(s) ds$ . Two functions  $\psi$  and  $\theta$  are of the same order if

$$\limsup_{t \rightarrow \infty} \psi(t)/\theta(t) < \infty, \quad \limsup_{t \rightarrow \infty} \theta(t)/\psi(t) < \infty.$$

This implies that  $E[\psi(Y)] < \infty$  if and only if  $E[\theta(Y)] < \infty$  for any nonnegative random variable  $Y$ .

Let  $\Psi_0$  be the class of all concave non-decreasing  $\psi$  with  $\psi(0) = 0$ ;  $\Phi_0$  the class of all convex  $\psi$  satisfying  $\psi(2t) \leq c\psi(t)$  for some  $c < \infty$  and  $\psi = \tilde{\theta}$  where  $\theta(0) = 0$  and  $\theta(t) \uparrow \infty$  as  $t \rightarrow \infty$ . Throughout the paper, let  $\varphi$  be a function of the same order as  $t \rightarrow t^m \varphi_0(t)$  where  $m$  is a nonnegative integer and  $\varphi_0 \in \Psi_0$ . In this case, we write  $\varphi(t) \simeq t^m \varphi_0(t)$ . If we define  $\varphi_j$  recursively by  $\varphi_j = \tilde{\varphi}_{j-1}$ ,  $j \geq 1$ , then  $\varphi$  is also of the same

order as  $\varphi_m$  (see Lemma 1 (b) of [10]).

Let  $Y_n^i = (\xi_n^i, \eta_n^i)$ ,  $a^i = E(\xi_n^i)$  and  $b^i = E(\eta_n^i)$  for  $i = 1, 2, 3$ . Throughout the paper, we make use of the following assumptions with no explicit mention of them. Malyshev [4] introduced Assumption 2, under which he proved ergodicity of the Markov chain (1.1) in the case of bounded jumps (see also [12]).

Assumption 1.  $E(|Y_1^i|) < \infty$  for  $i = 0, 1, 2, 3$ .

Assumption 2. Suppose that  $(a^3)^2 + (b^3)^2 > 0$  and one of the following conditions hold:

$$(2.6) \quad a^3 \geq 0, \quad b^3 < 0, \quad a^3 b^1 - b^3 a^1 < 0$$

$$\text{and } (a^2, b^2) \neq (0, b^2) \text{ with } b^2 \geq 0,$$

$$(2.7) \quad a^3 < 0, \quad b^3 \geq 0, \quad b^3 a^2 - a^3 b^2 < 0$$

$$\text{and } (a^1, b^1) \neq (a^1, 0) \text{ with } a^1 \geq 0,$$

$$(2.8) \quad a^3 < 0, \quad b^3 < 0, \quad a^3 b^1 - b^3 a^1 < 0 \text{ and } b^3 a^2 - a^3 b^2 < 0.$$

We write  $E_x$  and  $P_x$  for expectation and probability conditional on  $X_0 = x$  respectively. For the inner product between the vectors  $x$  and  $y$  in  $N \times N$ , we make use of the notation  $x \cdot y$ . The next proposition obtained by Vaninskii and Lazareva [12] is the most important tool for our rate of convergence results.

Proposition 2 (Vaninskii and Lazareva [12]). Under Assumptions 1 and 2, there exist a finite subset  $K$  of  $N \times N$ , a vector  $v = (v^1, v^2)$  with  $v^1 > 0$  and  $v^2 > 0$ , and an  $N^*$ -valued function  $n(x)$  of  $x \in N \times N$  such that  $n(x) = n_i$  for  $x \in D_i$  and the inequality

(2.9)  $E_x\{v \cdot (Z_{n(x)} - x)\} \leq -\varepsilon$  for any  $x \in (N \times N) \setminus K$   
holds for some  $\varepsilon > 0$ .

3. The rate of convergence. In this section, we obtain the rate of convergence results under Assumptions 1 and 2. For this purpose, we make use of an embedded chain  $\{\hat{Z}_n\}$  which will be constructed as in §19.1.1 of [6]. Let  $s(k)$  be defined by  $s(0) = 0$  and

$$(3.1) \quad s(k+1) = s(k) + n(Z_{s(k)}), \quad k = 0, 1, 2, \dots$$

Then, by virtue of the strong Markov property, we obtain that

$$(3.2) \quad \hat{Z}_k := Z_{s(k)}, \quad k = 0, 1, 2, \dots$$

is a Markov chain with transition law  $\hat{P}$ , where

$$(3.3) \quad \hat{P}(x, A) = P^{n(x)}(x, A), \quad x \in N \times N, \quad A \subset N \times N.$$

We let  $\tau_A$  (resp.  $\hat{\tau}_A$ ) denote the first hitting time on a set  $A$  for the Markov chain  $\{Z_n\}$  (resp.  $\{\hat{Z}_n\}$ ).

When  $Z_0 = x$ , we denote by  $U_x$  the increment of  $\{\hat{Z}_n\}$  to be added at time 1:

$$(3.4) \quad U_x = \hat{Z}_1 - x = Z_{n(x)} - x = \sum_{j=1}^{n(x)} \sum_{i=0}^3 I(Z_{j-1} \in D_i) Y_j^i.$$

Then, the inequality (2.9) can be rewritten in the following form:

For some  $\varepsilon > 0$ ,

$$(3.5) \quad E_x(v \cdot U_x) \leq -\varepsilon \quad \text{for any } x \in (N \times N) \setminus K$$

holds. We define the positive integer  $\hat{n}$  by  $\hat{n} = \max\{n_0, n_1, n_2, n_3\}$ . Making use of (3.5), we can show ergodicity and geometric ergodicity as follows.

**Theorem 1.** (1) (Vaninskii and Lazareva [12])  $\{Z_n\}$  is ergodic.

(2) If, for some  $\lambda > 0$ ,  $E[\exp(\lambda |Y_1^i|)] < \infty$ ,  $i = 0, 1, 2, 3$ , then

$\{Z_n\}$  is also geometrically ergodic.

Remark 1. When Assumption 2 holds, the conclusion of (1) was proved by several authors: e.g., Malyshev [4], Malyshev and Menshikov [5], Rosenkrantz [8] and Fayolle [2] under various stronger assumptions than Assumption 1. The conclusion of (2) was proved by Malyshev and Menshikov [5] in the case of bounded jumps.

We now turn to discuss sub-geometric rates of convergence of  $\{Z_n\}$  by making use of the approach developed by Tweedie [11] and Thorisson [10]. The next lemma readily follows from definition of  $U_x$  and Assumption 1.

Lemma 1. If  $\psi \in \Psi_0 \cup \Phi_0$  and  $E[\psi(|Y_1^i|)] < \infty$ ,  $i = 0, 1, 2, 3$ , then  $\sup_x E_x[\psi(|U_x|)] < \infty$ .

The next lemma is adapted for the random walk (1.1) from Lemma 4 of [10] for the one-dimensional random walk on  $R_+^1$ .

Lemma 2. Suppose that  $\psi \in \Psi_0 \cup \Phi_0$ ,  $E\{|Y_1^i| \psi(|Y_1^i|)\} < \infty$ ,  $i = 0, 1, 2, 3$ , and that there exist  $c > 0$  and a finite set  $\tilde{K} \supset K$  such that

$$(3.6) \quad E_x\{\psi(\tau_{\tilde{K}})\} \leq c \psi(v \cdot x) \quad \text{for all large enough } x.$$

Then the inequality (3.6) holds with  $\psi$  replaced by  $\tilde{\psi}$  and  $\tilde{K}$  replaced by a finite set  $\tilde{\tilde{K}} \supset \tilde{K}$ .



Proposition 3. If  $E\{\varphi(|Y_1^i|)\} < \infty$ ,  $i = 0, 1, 2, 3$ , then there exists a finite set  $L \supset K$  such that

$$(3.7) \quad E_x\{\varphi(\hat{\tau}_L)\} < c_1 \varphi(v \cdot x), \quad x \in (N \times N) \setminus L$$

holds for some  $c_1 > 0$ , and hence

$$(3.8) \quad E_x\{\varphi(\tau_L)\} < c_2 \varphi(v \cdot x), \quad x \in (N \times N) \setminus L$$

holds for some  $c_2 > 0$ .

By virtue of Proposition 3, we can easily obtain the following main result.

Theorem 2. If  $E\{\tilde{\varphi}(|Y_1^i|)\} < \infty$ ,  $i = 0, 1, 2, 3$ , then  $\{Z_n\}$  is ergodic of order  $\varphi$ .

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